# Use of the Autocorrelation Function for Frequency Stability Analysis

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## Introduction

This paper describes the use of the autocorrelation function (ACF) as a complement to other statistical and spectral methods for frequency stability analysis.

## The Autocorrelation Function

The autocorrelation function of a time series z for lag k is defined as:

$$\rho_k = \mathrm{E}\left\{(z_t - \mu)(z_{t+k} - \mu)\right\} / \sigma^2$$

where: E{} is the expectation operator, $z_t$ is value of the time series at time t, μ is its mean, and $\sigma^2$ is its variance. A common estimate of the autocorrelation function is:

$$r_k = \frac{\frac{1}{N}\sum_{t=1}^{N-k}(z_t - \bar{z})(z_{t+k} - \bar{z})}{\frac{1}{N}\sum_{t=1}^{N}(z_t - \bar{z})^2}$$

where $\bar{z} = \frac{1}{N}\sum_{t=1}^{N} z_t$, and where the lags are k=0, 1, 1 … K, and K is ≤ N-1.

An equivalent (and faster) estimate can be made for the summation in the numerator as the product of the Fourier transforms of the two terms, based on the fact that convolution in the time domain is equivalent to multiplication in the frequency domain. The autocorrelation sequence calculated using the Fast Fourier Transform (FFT) produces autocorrelation points at lags up to one-half of the data length. An autocorrelation plot is often restricted to fewer points to better show values at smaller lags.

## Uses of the Autocorrelation Function

Because the ACF and the power spectrum are related by the Fourier transform, they are mathematically equivalent, and contain the same information. However, the power spectrum is more familiar and its interpretation is generally easier. The autocorrelation sequence is most useful for theoretical work, for determining the non-whiteness of data or residuals, for detecting periodic components in data, and for identifying the dominant power law noise type. The latter technique is the main subject of this paper.

## Examples of Autocorrelation Plots for Power Law Noise

Figure 1 shows autocorrelation plots, along with the corresponding data, for white, flicker and random walk noise. It is clear that there is a large difference in the degree of correlation between these data as indicated by the shape of their autocorrelations.

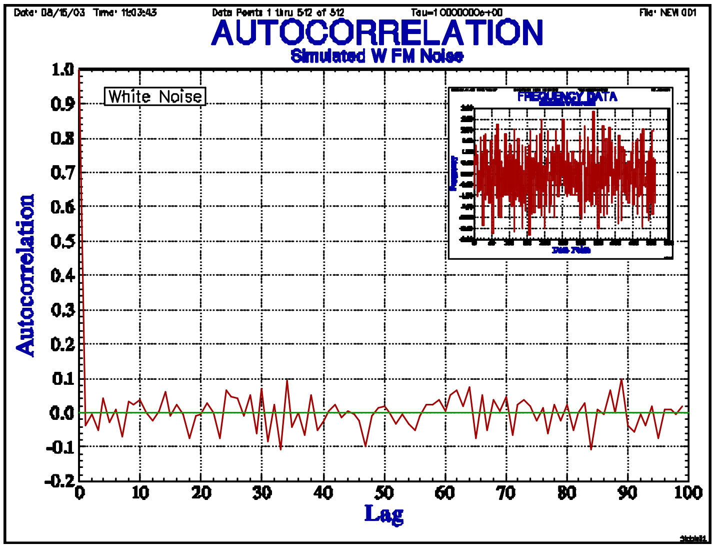


Figure 1a. White FM Noise Autocorrelation Plot

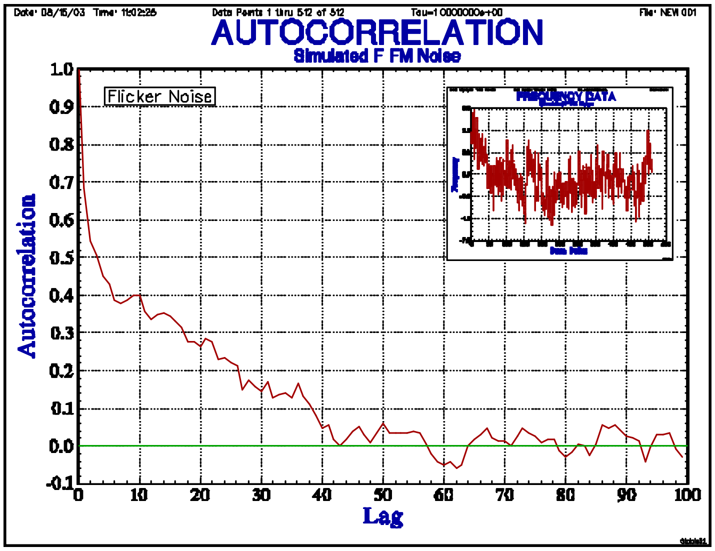


Figure 1b. Flicker FM Noise Autocorrelation Plot

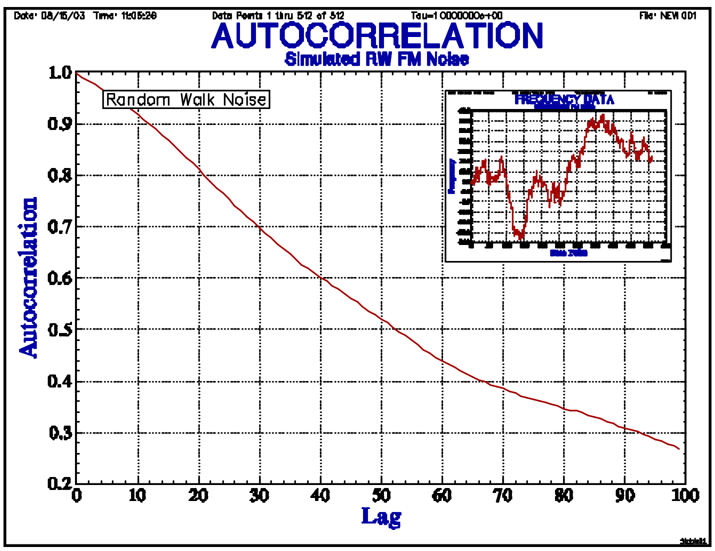


Figure 1c. Random Walk FM Noise Autocorrelation Plot

## · Lag 1 Scatter Plots

A Lag 1 scatter plot is a plot of the phase or frequency data plotted against itself with a lag of 1. The data at time t+1 is plotted on the y-axis versus the value at time t on the x-axis. This plot is another way of showing the degree of correlation in the data, and the slope of a linear fit to these points is closely related to the lag 1 autocorrelation. Examples of Lag 1 scatter plots for frequency data having the five most common power law noise types are shown in Figure 2 below.

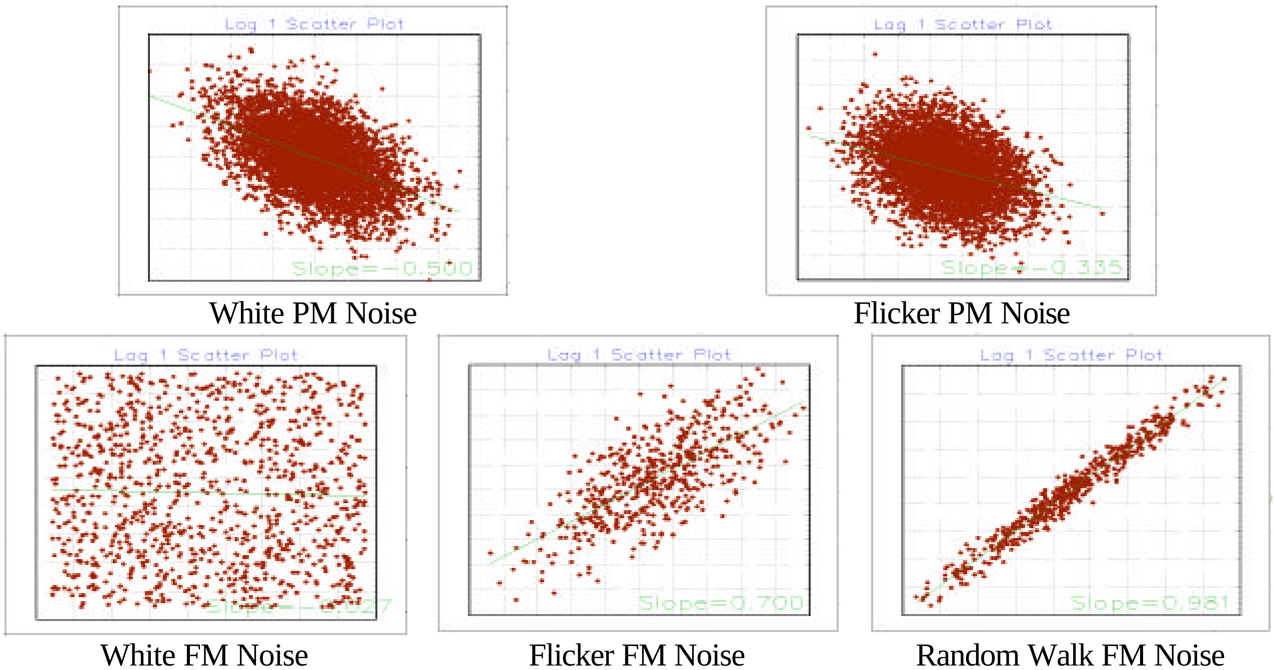


Figure 2. Lag 1 Scatter Plots for Frequency Data

## · Power Law Noise Identification

It is possible to identify the dominant power law noise process in a phase or frequency data record by examination of its autocorrelation function. In particular, the lag 1 autocorrelation can be used for that purpose, as shown in the following plot.

The Stable32 Autocorrelation function provides an estimate of the power law noise type (white, flicker, random walk, flicker walk and random run) for the particular data type (phase or frequency. This estimate is based on the lag 1 autocorrelation value, and is effective for data sets of 30 points and larger. White (uncorrelated) noise has a nominal lag 1 autocorrelation of zero, while flicker and white noise have nominal lag 1 autocorrelations of –1/3 and –1/2 respectively. The more divergent noises have lag 1 autocorrelation values that are positive, dependent on the number of samples, and tend to be large (approaching 1). For frequency data, the threshold values that separate the white, flicker and random walk FM power law noise types can be determined by empirical fits of the form $a + b \cdot \ln(N)$, where N is the # of analysis points, to the lag 1 values for equal amounts of the two adjacent noise types. Those lines have the same log slope separated by 0.5, with the flicker - random walk noise boundary is limited to a maximum value of 0.998. The flicker and white PM noises have boundary values of –0.17 and –0.42 that are independent of N. These noise regions are shown in Figure 3 below. It is difficult, however, to distinguish the more divergent noise processes by this method (e.g. flicker walk FM and random run FM for frequency data, and flicker FM and above (lower $\alpha$) for phase data.

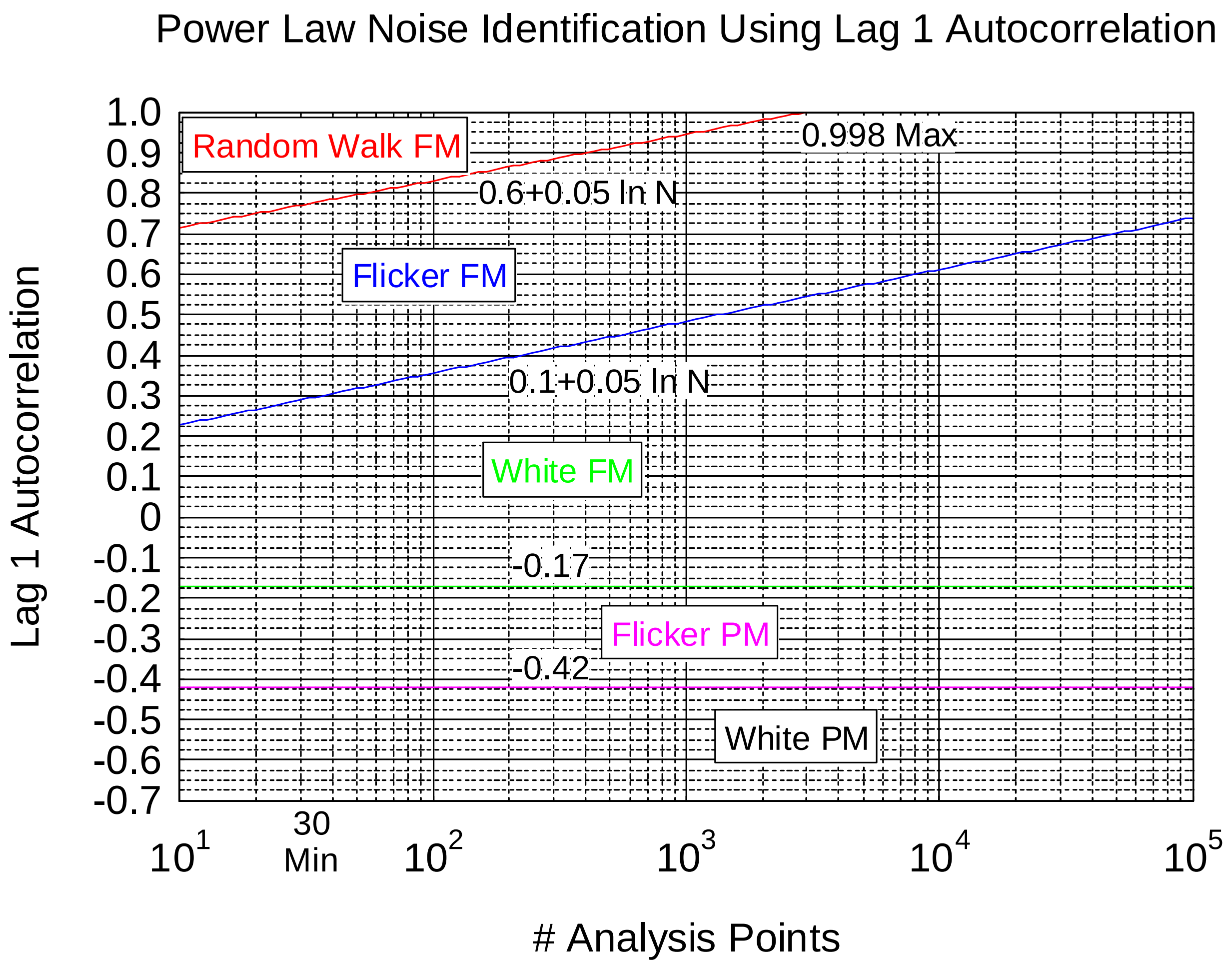


Figure 3. Lag 1 Autocorrelation Power Law Noise Boundaries for Frequency Data

A more refined method for identifying power law noises using the lag 1 autocorrelation has been suggested by C. Greenhall [4] based on the properties of fractionally integrated noises having spectral densities of the form $f^{-2d}$. For δ < ½, the process is stationary and has a lag 1 autocorrelation that is independent of N equal to $r_1 = \frac{d}{1-d}$, and the noise type can therefore be estimated from $d = \frac{r_1}{1+r_1}$. For frequency data, white PM noise has $r_1 = -1/2$, flicker PM noise has $r_1 = -1/3$, and white FM noise has $r_1 = 0$. For the more divergent noises, first differences of the data are taken until a stationary process is obtained ($d < 0.25$). The noise identification process therefore uses $p = -round(2d) - 2d$, where $round(2d)$ is $2d$ rounded to the nearest integer and $d$ is the number of times that the data is differenced to bring δ down to < 0.25. If $z$ is a τ-average of frequency data y(t), then α = p; if $z$ is a τ-average of phase data x(t), then α = p + 2, where α is the usual power law exponent $f^{\alpha}$, thereby determining the noise type at that averaging time. This is the method used in Stable32 for estimating the noise type from the lag 1 autocorrelation. It has excellent discrimination for all common power law noises for both phase and frequency data, including difficult cases with mixed noises. We encourage you to experiment with it for your applications.

## Stable32 Autocorrelation Function

The Stable32 Autocorrelation function has provisions for plotting the ACF for a selectable number of lags after applying a chosen averaging factor to the phase or frequency data. It provides an estimate of the power law noise type (white, flicker, random walk, flicker walk, or random run) for the particular data type (phase or frequency) that is displayed as a message on the plot. This estimate includes both the name of the closest power law noise type and the estimated alpha value, based on the lag 1 autocorrelation value, and is available for data sets of 30 and larger. The estimated alpha value is particularly helpful for mixed noises because it indicates the approximate proportions of the two dominant noise types (e.g. alpha=1.50 indicates an equal mixture of white and flicker PM noise). Stable32 also has provisions for plotting a lag scatter plot for a selected lag number, both as an insert on the autocorrelation plot and as a separate plot that includes 1, 2 and 3 sigma error boxes. These lag scatter plots include least square linear fits to the slope of the plotted points. Typical Stable32 ACF and lag scatter plots are shown in Figure 4 below.

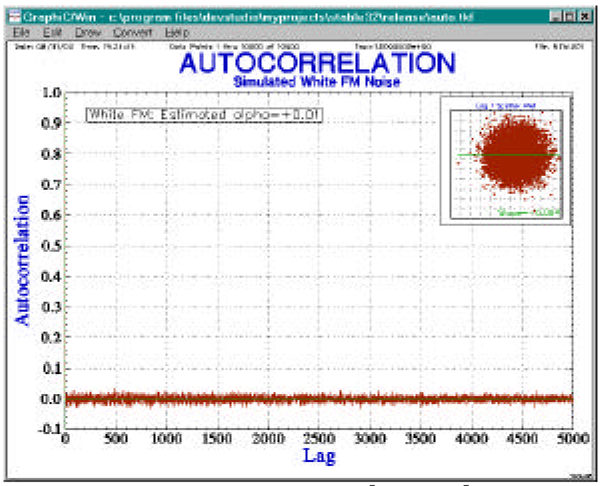


Figure 4a. Autocorrelation Plot

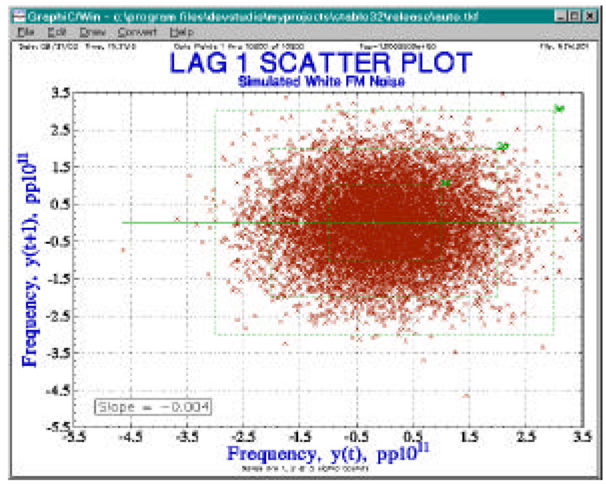


Figure 4b. Lag 1 Scatter Plot

## References

The following references apply to the use of the autocorrelation function in the Stable32 program for frequency stability analysis.


1. W.H. Press, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numercial Recipes in C*, Cambridge University Press, 1988, ISBN 0-521-35465-X, Section 13.2.
2. G. Box and G. Jenkins, *Time Series Analysis*, Holden-Day, Inc., 1976, ISBN 0-8162-1104-3.
3. D.B. Percival and A.T. Walden, *Spectral Analysis for Physical Applications*, Cambridge University Press, 1993, ISBN 0-521-43541-2.
4. C. Greenhall, "Another Power-Law Identifier That Uses Lag-1 Autocorrelation", private communication, August 28, 2003.


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